STABLE COMMUTATOR LENGTH OF A DEHN TWIST

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Abstract. It is proved that the stable commutator length of a Dehn twist in the mapping class group is positive and the tenth power of a Dehn twist about a nonseparating simple closed curve is a product of two commutators. As an application a new proof of the fact that the growth rate of a Dehn twist is linear is given.

0. Introduction

The purpose of this paper is to prove that in the mapping class group of an orientable surface the stable commutator length of a Dehn twist about a simple closed curve not bounding a disc with punctures is positive for every g. We also give an upper bound for it. This gives an asymptotic estimate to Problem 2.13 $(B)(C)(D)$ in Kirby's problem book [8].

The upper bound for the stable commutator length of a Dehn twist is given based on known results. In the nonseparating case, however, we get a better upper bound by proving that the tenth power of such a Dehn twist is a product of two commutators, which is an interesting result itself.

It was shown by Farb, Lubotzky and Minsky in [6] that the growth rate of a Dehn twist on an orientable surface of genus at least one is linear, answering a question of Ivanov (cf. Problem 2.16 in [8]). As an application of our main result we give a new proof of this fact by extending it to the genus zero case.

The positivity of the stable commutator length of a Dehn twist about a separating simple closed curve was proved by Endo and Kotschick in [5]. They concluded from this that the mapping class groups are not uniformly perfect and that the natural map from the second bounded cohomology to the ordinary cohomology of the mapping class group

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is not injective, which verified two conjectures of Morita. (cf. [14], Conjectures 6.19 and 6.21.) In the proof of their main result, they use Seiberg-Witten theory.

The main idea of our proof of the main result of this paper is to use the handlebody decomposition of a 4-manifold admitting a Lefschetz fibrations. This was suggested to the author by Stipsicz for a signature computation in [9]. In the computation we use the symplectic Parshin-Arakelov inequality of Li [12] to prove the nonexistence of certain Lefschetz fibrations.

Donaldson [3] proved that every symplectic 4-manifold admits a Lefschetz fibration after perhaps blowing up. Conversely, Gompf [7] showed that the total space of every genus-g Lefschetz fibration admits a symplectic structure provided $g \geq 2$. This gives a combinatorial approach to symplectic 4-manifolds through certain relations in mapping class groups. But understanding the relations in mapping class groups is not so easy. Usually, information in mapping class groups gives information about the corresponding 4-manifolds. Examples of such applications are given in [4, 10, 9]. The present paper, however, gives an application in the reverse direction.

This paper has grown from a question of András Stipsicz who asked to the author whether the mapping class groups were uniformly perfect. Author thanks him and Dieter Kotschick for their comments on the content of this paper, and the referee for his/her suggestions.

1. Preliminaries

For a compact orientable surface S of genus g with p marked points (to which we call punctures) and q boundary components, we denote by $\mathrm{Mod}_{g,p}^q$ the mapping class group of S, the group of isotopy classes of orientation-preserving diffeomorphisms $S \to S$ which restrict to the identity on the boundary and preserve the set of punctures. The isotopies are also assumed to be the identity on the boundary and punctures. If p and/or q is zero, we omit it from the notation, so that, for example, Mod_g denotes $\mathrm{Mod}_{g,0}^0$.

Let S be an oriented surface. A simple closed curve a on S is called *trivial* if it bounds either a disc or a disc with one puncture. For every simple closed curve a on S, there is a well known diffeomorphism called (right) Dehn twist about a, denoted by t_a, obtained by cutting the surface along a and twisting one of the side to the right by 2π and gluing the two side back. A diffeomorphism and its isotopy class are denoted by the same symbol, and similarly for simple closed curves. For a Dehn twist t_a, we always assume that a is nontrivial as the Dehn

twist about a trivial simple closed curve is itself trivial in the mapping class group.

We first state the next lemma which is elementary and will be used in the sequel. A proof of it can be found in [15], Theorem F, page xiii.

Lemma 1.1. *Let $\{a_n\}$ be a sequence of real numbers with nonnegative terms such that $a_{n+m} \leq a_n + a_m$ for every n and m. Then the limit $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$ exists.*

For a group G let $[G,G]$ denote the commutator subgroup, the subgroup of G generated by all commutators $[a,b] = aba^{-1}b^{-1}$ for $a, b \in G$. For $x \in [G,G]$, we define the commutator length $c(x)$ of x to be the minimum number of factors needed to express x as a product of commutators. Clearly, $c(x^{n+m}) \leq c(x^n) + c(x^m)$. Therefore, we can define

$$||x|| = \lim_{n\to\infty} \frac{c(x^n)}{n},$$

which is called the *stable commutator length of* x.

Recall that for a group G, the first homology group $H_1(G)$ of G with integer coefficients is isomorphic to the derived quotient group $G/[G,G]$.

The next theorem and the corollary will be useful for us.

Theorem 1.2 ([2])**.** *Let G be a group and let $u, v \in G$. Then $[u,v]^k$ can be written as a product of $E(\frac{k}{2}) + 1$ commutators, where $E(\frac{k}{2})$ denotes the integer part of $\frac{k}{2}$.*

Corollary 1.3 ([1])**.** *Let G be a group and let $u_1, v_1, u_2, v_2, \ldots u_r, v_r$ be elements of G. Then $([u_1,v_1][u_2,v_2]\ldots[u_r,v_r])^k$ is can be written as a product of $k(r-1) + E(\frac{k}{2}) + 1$ commutators.*

Proof. The proof follows from $(uv)^k = (uvu^{-1})(u^2vu^{-2})\ldots(u^kvu^{-k})u^k$ and Theorem 1.2. □

2. The main result

It is well known that the mapping class group Mod_g is perfect when $g \geq 3$ (cf. [16]). Hence, every element, in particular each Dehn twist, is a product of commutators. In the case of $g = 2$, $H_1(\mathrm{Mod}_2)$ is isomorphic the cyclic group of order 10 and is generated by the class of any Dehn twist about a nonseparating simple closed curve. A Dehn twist t_a in Mod_2 is not contained in the commutator subgroup, but t_a^{10} is. Hence, we can talk about $||t_a^{10}||$ in this case.

Our main result is the following theorem.

Theorem 2.1. *Let S be a closed connected oriented surface of genus $g \geq 2$ and let a be a nontrivial simple closed curve on S. Then $||t_a|| \geq \frac{1}{18g-6}$ if $g \geq 3$ and $||t_a^{10}|| \geq \frac{1}{3}$ if $g = 2$.*

Proof. Suppose first that $g \geq 3$. We assume the contrary that $||t_a|| < \frac{1}{18g-6}$. Choose a rational number r with $||t_a|| < r < \frac{1}{18g-6}$. Then there exists an arbitrarily large positive integer n such that rn is an integer and t_a^n can be written as a product of rn commutators. This gives a relatively minimal genus-g Lefschetz fibration over a closed orientable surface Σ of genus rn with the vanishing cycle a repeated n times as follows (We refer the reader to [7] for the details of the theory of Lefschetz fibrations). Consider $D^2 \times S$, where D^2 is the 2-disc. Attach n 2-handles to $\partial D^2 \times S$ along the simple closed curve a with -1 framing relative to the product framing. This gives a relatively minimal genus-g Lefschetz fibration $X_1 \to D^2$ with monodromy t_a^n along the boundary ∂D^2. Since t_a^n is a product of rn commutators, there is a surface bundle X_2 with fibers S over an orientable surface of genus rn with one boundary component such that the monodromy along the boundary is t_a^n. The boundary of X_1 and X_2 are genus-g surface bundles over S^1 with monodromy t_a^n. Now glue X_1 and X_2 via an fiber preserving orientation reversing diffeomorphism between boundaries to get a relatively minimal Lefschetz fibration $X \to \Sigma$ over a closed connected orientable surface Σ with the generic fiber S.

The Euler characteristic of X is easily computed to be

$$\chi(X) = 4(g-1)(rn-1) + n = 4grn - 4rn - 4g + 4 + n\ .$$

Also, it follows from the construction that $b_1(X) \leq 2g + 2rn$.

We now give a lower bound for $b_2^-(X)$. For each $i = 1, 2, \ldots, n-1$, the cores of the ith and $(i+1)$st 2-handles attached along a give a sphere S_i whose self intersection is -2. If $[S_i]$ denotes the homology class in $H_2(X;\mathbb{R})$ of S_i, then $[S_i][S_{i+1}] = \pm 1$ and $[S_i][S_j] = 0$ for $|i-j| \geq 2$, since the attaching regions of 2-handles can be chosen to be disjoint. We can orient S_i so that $[S_i][S_{i+1}] = -1$. It follows that the homology classes $[S_1], \ldots, [S_{n-1}]$ are linearly independent and form a basis for an $n-1$ dimensional subspace V of $H_2(X;\mathbb{R})$. Hence, the matrix of the intersection form restricted to V in the above basis is the matrix $-A$, where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$$

It is easy to check that the matrix A is positive definite. Therefore, the restriction of the intersection form to V is negative definite. It follows that

$$b_2^-(X) \geq n-1.$$

An easy computation gives an upper bound for $b_2^+(X)$;

$$\begin{aligned} n-1+b_2^+(X) &\leq b_2(X) \\ &= \chi(X)+2b_1(X)-2 \\ &\leq 4grn-4rn-4g+4+n+2(2g+2rn)-2 \\ &\leq 4grn+n+2. \end{aligned}$$

Hence,

$$b_2^+(X) \leq 4grn+3.$$

From these inequalities we obtain an upper bound for the signature of X;

$$\sigma(X) \leq 4grn-n+4.$$

On the other hand, following an argument of Kotschick [11], Li proved in [12] that

$$2(g-1)(rn-1) \leq c_1^2(X).$$

Hence, we obtain

$$\begin{aligned} 2(g-1)(rn-1) &\leq c_1^2(X) \\ &= 3\sigma(X)+2\chi(X) \\ &\leq 3(4grn-n+4)+2(4grn-4rn-4g+4+n) \\ &= 20grn-8rn-n+20-8g. \end{aligned}$$

As a result of this, we conclude that there exists arbitrarily big n such that

$$0 \leq [(18g-6)r-1]n+18-6g.$$

Since $(18g-6)r-1$ is negative, this is a contradiction.

This proves the theorem for $g \geq 3$.

If S is a closed surface of genus two, then taking n as a multiple of 10 above finishes the proof. □

Remark 2.2. I had originally proved that the stable commutator length of a Dehn twist in the above theorem are greater than or equal to $\frac{1}{20g-8}$, by using the inequality $c_1^2(X) \geq 0$ proved in [17]. The improvement was kindly suggested by Kotschick and Stipsicz.

Corollary 2.3. *Let S be a connected orientable surface of genus $g \geq 0$ with p punctures and q boundary components such that $g + q \geq 2$. Let a be a simple closed curve on S not bounding a disc with punctures. Suppose that t_a^k is in the commutator subgroup of* $\mathrm{Mod}_{g,p}^q$. *Then* $||t_a^k|| > 0$.

Proof. Let us glue a torus with one boundary component along each boundary component of S. By forgetting the punctures, we get a closed surface R of genus $g + q$. The circle a is now nontrivial on R. In this way we have a map F from the mapping class group of S to that of R. Clearly, $c(t_a^k) \geq c(F(t_a)^k)$. Since $F(t_a)$ is a Dehn twist about the nontrivial simple closed curve a on R, the corollary follows from $||t_a^k|| \geq ||F(t_a)^k||$ and Theorem 2.1. □

It was shown in [10] that the commutator length of a Dehn twist is two. Theorem 2.1 and Theorem 1.2 give the corollary.

Corollary 2.4. *Let S be a closed orientable surface of genus $g \geq 2$ and a be a nontrivial simple closed curve on S. Then the element t_a^k of* Mod_g *cannot be a commutator if $k > 9g - 3$.*

Proof. Assume that t_a^k is a commutator. Then t_a^{kn} is a product of $E(\frac{n}{2}) + 1$ commutators, i.e. $c(t_a^{nk}) \leq E(\frac{n}{2}) + 1$. Dividing both sides by kn and taking the limit as n tends to the infinity gives the desired contradiction $k \leq 9g - 3$. □

3. An upper bound for $||t_a||$.

In this section we give an upper bound for $||t_a||$ for a simple closed curve a. In the case a is nonseparating, we obtain a better upper bound by proving that the tenth power t_a^{10} of a Dehn twist t_a is a product of two commutators.

The next lemma is well known.

Lemma 3.1. *Let a and b be two simple closed curves on an oriented surface S.*

(a) *If a is disjoint from b, then t_a commutes with t_b.*
(b) *If a intersects b transversely at only one point, then $t_a t_b t_a = t_b t_a t_b$.*

Lemma 3.2. *Let S be a connected oriented surface and let a, b, c and d be four simple closed curves on S such that there is an orientation preserving diffeomorphism of S mapping a and b to d and c respectively. Then $t_a t_b^{-1} t_c t_d^{-1}$ is a commutator.*

Proof. Let g be the isotopy class of a diffeomorphism mapping a and b to d and c. Then

$$t_a t_b^{-1} t_c t_d^{-1} = t_a t_b^{-1} t_{g(b)} t_{g(a)}^{-1} = t_a t_b^{-1} g t_b t_a^{-1} g^{-1} = [t_a t_b^{-1}, g] \ .$$

□

Theorem 3.3. *Let S be a connected oriented surface of genus at least two and let a be a nonseparating simple closed curve on S. Then t_a^{10} can be written as a product of two commutators.*

Proof. Since Dehn twists about two nonseparating simple closed curves are conjugate and a conjugate of a commutator is again a commutator, it suffices to prove the theorem for some nonseparating simple closed curve.

Let a_1, a_2, a_3 be three nonseparating simple closed curves on S such that a_2 intersects a_1 and a_3 transversely only once, a_1 is disjoint from a_3 and $a_1 \cup a_3$ does not disconnect S. A regular neighborhood $a_1 \cup a_2 \cup a_3$ is a torus with two boundary components, say a_4 and a_5, which are nonseparating on S. Clearly, a_4 and a_5 are disjoint from a_1, a_2, a_3, and from each other. Let us denote t_{a_i} by t_i. It is well known that $t_4 t_5 = (t_1 t_2 t_3)^4$. Using Lemma 3.1, we obtain

$$\begin{aligned} t_4 t_5 &= (t_1 t_2 t_3)(t_1 t_2 t_3)(t_1 t_2 t_3)(t_1 t_2 t_3) \\ &= (t_1 t_2 t_1)(t_3 t_2 t_3)(t_1 t_2 t_1)(t_3 t_2 t_3) \\ &= (t_2 t_1 t_2)(t_2 t_3 t_2)(t_2 t_1 t_2)(t_2 t_3 t_2) \end{aligned}$$

Conjugating with t_2^{-1} gives

$$\begin{aligned} t_4 t_5 &= t_1 t_2 t_2 t_3 t_2 t_2 t_1 t_2 t_2 t_3 t_2 t_2 \\ &= t_1 (t_2^2 t_3 t_2^{-2}) t_2^4 t_1 t_2^{-1} (t_2^3 t_3 t_2^{-3}) t_2^{-1} t_2^6 \ . \end{aligned}$$

If we let $\alpha = t_2^2(a_3)$ and $\beta = t_2^3(a_3)$, we get the equality

$$(t_4 t_\alpha^{-1} t_5 t_1^{-1}) = t_2^4 (t_1 t_2^{-1} t_\beta t_2^{-1}) t_2^6 \ .$$

The curves a_4 and a_5 do not intersect α and a_1. Since the complements of $a_4 \cup \alpha$ and $a_5 \cup a_1$ are connected, there is an orientation preserving diffeomorphism taking a_4 and α to a_1 and a_5 respectively. The curve a_2 intersects a_1 and β transversely at one point. Hence, there is an orientation preserving diffeomorphism mapping a_1 and a_2 to a_2 and β respectively. By Lemma 3.2, each parenthesis is a commutator. Since the conjugate of a commutator is again a commutator, the proof follows. □

Theorem 3.4. *Let S be a connected oriented surface of genus $g \geq 2$ and let a be a simple closed curve on S.*

(a) *If a is nonseparating, then $||t_a|| \leq \frac{3}{20}$ when $g \geq 3$ and $||t_a^{10}|| \leq \frac{3}{2}$ when $g = 2$.*
(b) *If a is separating, then $||t_a|| \leq \frac{3}{4}$ when $g \geq 3$.*

Proof. If a is nonseparating, by Theorem 3.3 and Corollary 1.3, the element t_a^{10n} can be written as a product of $n + E(\frac{n}{2}) + 1$ commutators. The proof (a) follows from this.

If a is separating, then t_a^2 is a product of two commutators [10]. It follows now from Corollary 1.3 that $||t_a|| \leq \frac{3}{4}$. □

Remark 3.5. In the case of an oriented surface of genus 2, if a is a separating simple closed curve such that one of the components of the complement of a has genus at two, then it is easy to conclude from the proof of Proposition 10 in [10] that t_a^2 can be written a product of two commutators. It follows that $||t_a|| \leq \frac{3}{4}$.

If the genera of both components of the complement of a are one, then t_a^5 is in the commutator subgroup of Mod_2. There are two nonseparating simple closed curves b, c on S intersecting each other at one point such that $t_a = (t_b t_c)^6$. It can be shown from this that t_a^5 is a product of 21 commutators. It follows that $||t_a^5|| \leq \frac{41}{2}$.

4. An application: Growth rate of Dehn twists

It is well known that the mapping class groups are finitely presented (cf. [18]). For a finite generating set A of $\mathrm{Mod}_{g,p}^q$, let d denote the corresponding word metric on $\mathrm{Mod}_{g,p}^q$. That is, if $f, g \in \mathrm{Mod}_{g,p}^q$ then

$$d(f, g) = \min\{n \mid g^{-1}f = x_1 x_2 \cdots x_n, x_i \in A\}.$$

In [6], Farb, Lubotzky and Minsky proved that if $g \geq 1$ then Dehn twists in the mapping class group $\mathrm{Mod}_{g,p}^q$ have linear growth rate. That is, the limit

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n}$$

is positive. Here, we give another proof of this fact and extend it to the genus zero case. Note that this limit does depend on the choice of the generating set, but the positivity of it does not.

Theorem 4.1. *Let S be a connected oriented surface of genus g with p punctures and q boundary components. Suppose that $g + q \geq 2$. If a is a simple closed curve on S not bounding a disc with punctures, then*

for any finite generating set of the mapping class group, the limit

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n}$$

is positive, i.e. the growth rate of $d(t_a^n, 1)$ is linear.

Proof. Suppose first that S is a closed surface so that $g \geq 2$. Let A be a finite generating set for the mapping class group Mod_g. In the case of $g = 2$, we can choose A so that it contains a set of generators for the commutator subgroup of Mod_2. Let k be a positive integer such that each element of A contained in the commutator subgroup can be written as a product of k commutators. Thus, t_a^{10n} can be written as a product of $k\, d(t_a^{10n}, 1)$ commutators for any positive integer n. Hence $c(t_a^{10n}) \leq k d(t_a^{10n}, 1)$. It follows that

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n} \geq \frac{1}{10k}\, ||t_a^{10}||.$$

Hence, it is positive by Theorem 2.1.

In the general case, let us glue a torus with one boundary to S along each boundary component of S and forget the punctures. Let A be a finite generating set for $\mathrm{Mod}_{g,p}^q$. As in the proof of Corollary 2.3, this gives a homomorphism $F : \mathrm{Mod}_{g,p}^q \to \mathrm{Mod}_{g+q}$. Extend $F(A)$ to a finite generating set B for Mod_{g+q}. Clearly, we have $d(t_a^n, 1) \geq d(F(t_a)^n, 1)$ with respect to the generating sets A and B. Since $F(t_a)$ is a Dehn twist about a nontrivial simple closed curve on a closed surface, the proof follows from the closed case. □

5. A question

We end with a question, which arises from the topology of the Stein fillings of a contact 3-manifold: Let S be an oriented surface with one boundary component and let $a_1, a_2, \ldots$ be a sequence of nonseparating simple closed curves on S. Does there exist a positive integer N such that

$$c(t_{a_1} t_{a_2} \cdots t_{a_n}) \geq Nn.$$

More generally, does the limit

$$\lim_{n\to\infty} \frac{c(t_{a_1} t_{a_2} \cdots t_{a_n})}{n}$$

exist? If it does, is it positive (for any choice of a_i)? Note that if all a_n are equal, then we proved that there exists such an N.

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